

# Non-Abelian Superconductivity

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We establish a non-Abelian superconductivity and a non-Abelian Meissner effect by constructing an effective field theory of superconductivity in which a genuine  $SU(2)$  gauge symmetry governs the dynamics. We show that the magnetic flux is quantized in the unit of  $4\pi/g$ , not  $2\pi/g$ , in the non-Abelian superconductor.

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The superconductivity has always been based on quantum electrodynamics which is an Abelian gauge theory. The only exception is the  $SO(5)$  model of high- $T_c$  superconductivity [1]. In this view one might assume that the underlying dynamics of the recently discovered two-gap superconductor made of  $MgB_2$  [2] should also be the Abelian (i.e., electromagnetic) interaction. An important feature of multi-gap superconductors, however, is its non-Abelian structure, since multi-gap superconductors can be described by multi-component condensates which can naturally form a non-Abelian multiplet [3]. This raises the possibility of a non-Abelian superconductivity and a non-Abelian Meissner effect.

The motivation for a non-Abelian superconductivity must be clear. Suppose the underlying dynamics of the two-gap superconductor is the Abelian interaction. If so, the effective theory of two-gap superconductor should be an Abelian gauge theory which has a global  $SU(2)$  symmetry. But in this case the two condensates must carry the same charge (two electron-electron pairs or two hole-hole pairs), because there is no way that the Abelian gauge field can couple to a doublet condensate whose components have opposite charges (one electron-electron pair and one hole-hole pair). This means that a two-gap superconductor made of a doublet whose components have opposite charges can not be described by an Abelian gauge theory (unless the two components do not form a doublet). This poses a problem because this implies that one can not construct a theory of two-gap superconductor made of oppositely charged condensates based on an Abelian gauge theory.

*The purpose of this paper is to establish a non-Abelian superconductivity and a non-Abelian Meissner effect by constructing a non-Abelian gauge theory of superconductivity in which the magnetic flux is quantized in the unit*

$4\pi/g$ . We present an  $SU(2)$  gauge theory of superconductivity which is governed by a genuine non-Abelian dynamics, which can describe a two-gap superconductor made of an oppositely charged doublet condensate. As far as we understand, this type of non-Abelian superconductivity has never been discussed before.

To understand the non-Abelian superconductivity we must understand the Abelian two-gap superconductor first, because the non-Abelian superconductor is closely related to the Abelian two-gap superconductor. So we start with the Abelian two-gap superconductor and establish the Meissner effect in the theory first.

Consider a charged doublet scalar field  $\phi$  coupled to the electromagnetic field

$$\mathcal{L} = -|D_\mu \phi|^2 + \mu^2 \phi^\dagger \phi - \frac{\lambda}{2} (\phi^\dagger \phi)^2 - \frac{1}{4} F_{\mu\nu}^2, \quad (1)$$

where  $D_\mu \phi = (\partial_\mu + igA_\mu)\phi$ . This is an obvious generalization of the Landau-Ginzburg Lagrangian. The Lagrangian has the equation of motion

$$D^2 \phi = \lambda (\phi^\dagger \phi - \frac{\mu^2}{\lambda}) \phi, \\ \partial_\mu F_{\mu\nu} = j_\nu = ig [(D_\nu \phi)^\dagger \phi - \phi^\dagger (D_\nu \phi)], \quad (2)$$

which tells that a non-vanishing  $\langle \phi^\dagger \phi \rangle$  makes the photon massive. This implies the existence of Meissner effect. To demonstrate the Meissner effect we construct a magnetic vortex in this theory. Let

$$\phi = \frac{1}{\sqrt{2}} \rho \xi, \quad \xi^\dagger \xi = 1, \\ \rho = \rho(\varrho), \quad \xi = \begin{pmatrix} \cos \frac{f(\varrho)}{2} \exp(-im\varphi) \\ \sin \frac{f(\varrho)}{2} \end{pmatrix}, \\ A_\mu = \frac{m}{g} A(\varrho) \partial_\mu \varphi, \quad (3)$$

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and find that (2) is reduced to

$$\begin{aligned} \ddot{\rho} + \frac{1}{\varrho} \dot{\rho} - \left[ \frac{1}{4} \left( \dot{f}^2 + \frac{m^2}{\varrho^2} \sin^2 f \right) \right. \\ \left. + \frac{m^2}{\varrho^2} \left( A - \frac{\cos f + 1}{2} \right)^2 \right] \rho = \frac{\lambda}{2} (\rho^2 - \rho_0^2) \rho, \\ \ddot{f} + \left( \frac{1}{\varrho} + 2 \frac{\dot{\rho}}{\rho} \right) \dot{f} - 2 \frac{m^2}{\varrho^2} \left( A - \frac{1}{2} \right) \sin f = 0, \\ \ddot{A} - \frac{\dot{A}}{\varrho} - g^2 \rho^2 \left( A - \frac{\cos f + 1}{2} \right) = 0. \end{aligned} \quad (4)$$

Now, we impose the following boundary condition for the non-Abelian vortices [3],

$$\begin{aligned} \rho(0) = 0, \quad \rho(\infty) = \rho_0, \quad f(0) = \pi, \quad f(\infty) = 0, \\ A(0) = -1, \quad A(\infty) = 1. \end{aligned} \quad (5)$$

This need some explanation, because the boundary value  $A(0)$  is chosen to be  $-1$ , not  $0$ . This is to assure the smoothness of the scalar field  $\rho(\varrho)$  at the origin. Only with this boundary value  $\rho$  becomes analytic at the origin. One might object the boundary condition, because it creates an apparent singularity in the gauge potential at the origin. But notice that this singularity is an unphysical (coordinate) singularity which can easily be removed by a gauge transformation. In fact the singularity disappears with the gauge transformation

$$A_\mu \rightarrow A_\mu + \frac{m}{g} \partial_\mu \varphi, \quad (6)$$

which simultaneously changes the boundary condition  $A(0) = -1$ ,  $A(\infty) = 1$  to  $A(0) = 0$ ,  $A(\infty) = 2$ .

With the boundary condition we can integrate (4) and obtain the non-Abelian vortex solution of the two-gap superconductor, which is shown in Fig.1. Notice that the boundary condition (5) assures that the doublet  $\xi$  starts from the second component at the origin and ends up with the first component at the infinity, which shows that the non-Abelian vortex is different from the well-known Abrikosov vortex in ordinary (one-gap) superconductor [4].

Clearly the magnetic field  $H$  of the vortex has total flux given by

$$\hat{\phi} = \int H d^2x = \frac{2\pi m}{g} [A(\infty) - A(0)] = \frac{4\pi m}{g}. \quad (7)$$

Notice that the unit of the non-Abelian flux is  $4\pi/g$ , not  $2\pi/g$ . This is the non-Abelian quantization of magnetic flux that we have in two-gap superconductor. We emphasize that this is a direct consequence of the boundary condition  $A(0) = -1$  (or more precisely  $A(\infty) - A(0) = 2$ ) in (5).

But just as in ordinary superconductor this non-Abelian quantization of magnetic flux is of topological origin. To understand this notice that the doublet  $\xi$ ,

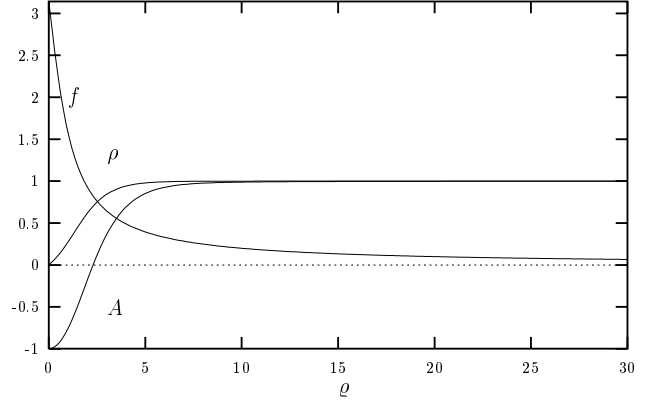


FIG. 1: The non-Abelian vortex with  $m = 1$  in two-gap superconductor. Here we have put  $g = \lambda = 1$ , and  $\varrho$  is in the unit of  $\rho_0$ .

with the  $U(1)$  gauge symmetry, forms a  $CP^1$  field. So it can naturally define a mapping  $\pi_2(S^2)$  from the compactified  $xy$ -plane  $S^2$  to the target space  $S^2$ , which can be classified by the flux quantum number

$$q = -\frac{1}{4\pi} \int \epsilon_{ij} \partial_i \xi^\dagger \partial_j \xi d^2x = m. \quad (8)$$

Again this non-Abelian topology should be compared with the well-known Abelian topology  $\pi_1(S^1)$  of the Abrikosov vortex.

The existence of the magnetic vortex solution demonstrates that the Lagrangian (1) generates the Meissner effect, and thus can describe the superconductivity in two-gap superconductor [3].

With the ansatz (3) one can express the Hamiltonian of the vortex as

$$\begin{aligned} \mathcal{H} = \frac{1}{2} \left[ \left( \dot{\rho} \pm \frac{m}{\varrho} \left( A - \frac{\cos f + 1}{2} \right) \rho \right)^2 \right. \\ \left. + \left( \dot{f} \pm \frac{m}{\varrho} \sin f \right)^2 \frac{\rho^2}{4} + \left( H \pm \frac{\sqrt{\lambda}}{2} (\rho^2 - \rho_0^2) \right)^2 \right. \\ \left. \pm H (g\rho^2 + \sqrt{\lambda}(\rho_0^2 - \rho^2)) \right], \end{aligned} \quad (9)$$

so that the Hamiltonian has a minimum value when

$$\begin{aligned} \dot{\rho} \pm \frac{m}{\varrho} \left( A - \frac{\cos f + 1}{2} \right) \rho &= 0, \\ \dot{f} \pm \frac{m}{\varrho} \sin f &= 0, \\ H \pm \frac{\sqrt{\lambda}}{2} (\rho^2 - \rho_0^2) &= 0. \end{aligned} \quad (10)$$

This can be viewed as a first order equation for the magnetic vortex. Indeed, when the coupling constant  $\lambda$  has the critical value (i.e., when  $\lambda = g^2$ ), one can integrate and reduce the second order equation (4) to the above first order equation.

Integrating the second equation of (10) we have

$$\cos f(\varrho) = \frac{\varrho^{2m} - a^2}{\varrho^{2m} + a^2}, \quad \sin f(\varrho) = \frac{2a\varrho^m}{\varrho^{2m} + a^2}, \quad (11)$$

where  $a$  is an integration constant, so that (10) is reduced to (with  $\lambda = g^2$ )

$$\begin{aligned} \dot{\rho} \pm \frac{m}{\varrho} \left( A - \frac{\varrho^{2m}}{\varrho^{2m} + a^2} \right) \rho &= 0, \\ \frac{m}{g} \frac{\dot{A}}{\varrho} \pm \frac{g}{2} (\rho^2 - \rho_0^2) &= 0. \end{aligned} \quad (12)$$

In this case the Hamiltonian becomes

$$\mathcal{H} = \frac{g}{2} H \rho_0^2, \quad (13)$$

and the energy (per unit length) acquires the absolute minimum value

$$E = 2\pi m \rho_0^2 = \frac{g}{2} \rho_0^2 \hat{\phi}. \quad (14)$$

This tells that the minimum energy is fixed by the topological flux quantum number.

This means that the non-Abelian vortex has twice as much magnetic flux and energy. Again this difference can be traced back to the boundary conditions (5). Mathematically this difference has a deep reason, which originates from the fact that the Abelian  $U(1)$  runs from 0 to  $2\pi$ , but the  $S^1$  fiber of  $SU(2)$  runs from 0 to  $4\pi$  [3].

With the above preliminaries, we now construct an effective theory of two-gap superconductor which is based on a  $SU(2)$  gauge theory. To do this we need to understand the mathematical structure of the  $SU(2)$  gauge theory better. A best way to do this is to start with the well-known decomposition of the gauge potential [5, 6]. Let  $\hat{n}$  be the gauge covariant unit triplet which select the charge direction in  $SU(2)$ , and decompose the non-Abelian gauge potential into the restricted potential  $\hat{A}_\mu$  and the valence potential  $\vec{X}_\mu$ ,

$$\begin{aligned} \vec{A}_\mu &= A_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n} + \vec{X}_\mu = \hat{A}_\mu + \vec{X}_\mu, \\ (A_\mu &= \hat{n} \cdot \vec{A}_\mu, \quad \hat{n}^2 = 1, \quad \hat{n} \cdot \vec{X}_\mu = 0), \end{aligned} \quad (15)$$

where  $A_\mu$  is the “electric” potential. Notice that the restricted potential is precisely the potential which leaves  $\hat{n}$  invariant under the parallel transport,

$$\hat{D}_\mu \hat{n} = \partial_\mu \hat{n} + g \hat{A}_\mu \times \hat{n} = 0. \quad (16)$$

Under the infinitesimal gauge transformation

$$\delta \hat{n} = -\vec{\alpha} \times \hat{n}, \quad \delta \vec{A}_\mu = \frac{1}{g} D_\mu \vec{\alpha}, \quad (17)$$

one has

$$\begin{aligned} \delta A_\mu &= \frac{1}{g} \hat{n} \cdot \partial_\mu \vec{\alpha}, \quad \delta \hat{A}_\mu = \frac{1}{g} \hat{D}_\mu \vec{\alpha}, \\ \delta \vec{X}_\mu &= -\vec{\alpha} \times \vec{X}_\mu. \end{aligned} \quad (18)$$

This tells that restricted potential  $\hat{A}_\mu$  by itself describes an  $SU(2)$  connection which enjoys the full  $SU(2)$  gauge degrees of freedom. Furthermore the valence potential  $\vec{X}_\mu$  forms a gauge covariant vector field under the gauge transformation. More importantly, the decomposition is gauge-independent. Once the gauge covariant topological field  $\hat{n}$  is given, the decomposition follows automatically independent of the choice of a gauge [5].

The importance of the decomposition (15) for our purpose is that we can construct a non-Abelian gauge theory which has a full non-Abelian gauge degrees of freedom, with the restricted potential  $\hat{A}_\mu$  alone [5]. This is because the valence potential  $\vec{X}_\mu$  can be treated as a gauge covariant source, so that one can always exclude it from the theory without compromising the gauge invariance. Indeed we will see that it is this restricted gauge theory which describes the non-Abelian superconductivity.

To demonstrate the non-Abelian superconductivity consider the Lagrangian which is made of two condensates which couple to the restricted  $SU(2)$  gauge field,

$$\mathcal{L} = -|\hat{D}_\mu \Phi|^2 + \mu^2 \Phi^\dagger \Phi - \frac{\lambda}{2} (\Phi^\dagger \Phi)^2 - \frac{1}{4} \hat{F}_{\mu\nu}^2, \quad (19)$$

where now

$$\hat{D}_\mu \Phi = (\partial_\mu + \frac{g}{2i} \vec{\sigma} \cdot \hat{A}_\mu) \Phi.$$

The equation of motion of the Lagrangian is given by

$$\begin{aligned} \hat{D}^2 \Phi &= \lambda (\Phi^\dagger \Phi - \frac{\mu^2}{\lambda}) \Phi, \\ \hat{D}_\mu \hat{F}_{\mu\nu} &= \vec{j}_\nu = g \left[ (\hat{D}_\nu \Phi)^\dagger \frac{\vec{\sigma}}{2i} \Phi - \Phi^\dagger \frac{\vec{\sigma}}{2i} (\hat{D}_\nu \Phi) \right]. \end{aligned} \quad (20)$$

Let  $\xi$  and  $\eta$  be two orthonormal doublets which form a basis,

$$\begin{aligned} \xi^\dagger \xi &= 1, \quad \eta^\dagger \eta = 1, \quad \xi^\dagger \eta = \eta^\dagger \xi = 0, \\ \xi^\dagger \vec{\sigma} \xi &= \hat{n}, \quad \eta^\dagger \vec{\sigma} \eta = -\hat{n}, \\ (\hat{n} \cdot \vec{\sigma}) \xi &= \xi, \quad (\hat{n} \cdot \vec{\sigma}) \eta = -\eta, \end{aligned} \quad (21)$$

and let

$$\Phi = \phi_+ \xi + \phi_- \eta, \quad (\phi_+ = \xi^\dagger \Phi, \quad \phi_- = \eta^\dagger \Phi). \quad (22)$$

Now, with the identity

$$\begin{aligned} \left[ \partial_\mu - \frac{g}{2i} (C_\mu \hat{n} + \frac{1}{g} \hat{n} \times \partial_\mu \hat{n}) \cdot \vec{\sigma} \right] \xi &= 0, \\ \left[ \partial_\mu + \frac{g}{2i} (C_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n}) \cdot \vec{\sigma} \right] \eta &= 0, \end{aligned} \quad (23)$$

we find

$$\hat{D}_\mu \Phi = (D_\mu \phi_+) \xi + (D_\mu \phi_-) \eta, \quad (24)$$

where

$$\begin{aligned} D_\mu \phi_+ &= (\partial_\mu + \frac{g}{2i} A_\mu) \phi_+, \quad D_\mu \phi_- = (\partial_\mu - \frac{g}{2i} A_\mu) \phi_-, \\ A_\mu &= A_\mu + C_\mu, \\ C_\mu &= \frac{2i}{g} \xi^\dagger \partial_\mu \xi = -\frac{2i}{g} \eta^\dagger \partial_\mu \eta. \end{aligned}$$

From this we can express (19) as

$$\begin{aligned}\mathcal{L} = & -|D_\mu\phi_+|^2 - |D_\mu\phi_-|^2 + m^2(\phi_+^\dagger\phi_+ + \phi_-^\dagger\phi_-) \\ & - \frac{\lambda}{2}(\phi_+^\dagger\phi_+ + \phi_-^\dagger\phi_-)^2 - \frac{1}{4}\mathcal{F}_{\mu\nu}^2, \\ \mathcal{F}_{\mu\nu} = & \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu.\end{aligned}\quad (25)$$

This tells that the restricted  $SU(2)$  gauge theory (19) is reduced to an Abelian gauge theory coupled to oppositely charged scalar fields  $\phi_+$  and  $\phi_-$ . We emphasize that this Abelianization is achieved without any gauge fixing.

The Abelianization assures that the non-Abelian theory is not different from the two-gap Abelian theory. Indeed with

$$\chi = \begin{pmatrix} \phi_+ \\ \phi_-^* \end{pmatrix}, \quad (26)$$

we can express the Lagrangian (25) as [7]

$$\begin{aligned}\mathcal{L} = & -|D_\mu\chi|^2 + \mu^2\chi^\dagger\chi - \frac{\lambda}{2}(\chi^\dagger\chi)^2 - \frac{1}{4}\mathcal{F}_{\mu\nu}^2, \\ D_\mu\chi = & (\partial_\mu + ig\mathcal{A}_\mu)\chi,\end{aligned}\quad (27)$$

This is formally identical to the Lagrangian (1) of two-gap Abelian superconductor. The only difference is that here  $\phi$  and  $A_\mu$  are replaced by  $\chi$  and  $\mathcal{A}_\mu$ . This establishes that, with the proper redefinition of field variables (22) and (24), our non-Abelian restricted gauge theory of superconductivity can in fact be made identical to the Abelian gauge theory of two-gap superconductor. This proves the existence of non-Abelian superconductors made of the doublet consisting of oppositely charged condensates. As importantly our analysis tells that the two-gap Abelian superconductor has a hidden non-Abelian gauge symmetry because it can be transformed to the non-Abelian restricted gauge theory. This implies that the underlying dynamics of two-gap superconductor is indeed the non-Abelian gauge symmetry. In the non-Abelian superconductor it is explicit. But in the two-gap Abelian superconductor it is hidden, where the full non-Abelian gauge symmetry only becomes transparent when one embeds the nontrivial topology properly into the non-Abelian symmetry [7].

Once the equivalence of the two Lagrangians (1) and (19) is established, it must be evident that our non-Abelian theory of superconductivity also allows the non-Abelian vortex which has the non-Abelian flux quantization, and thus the non-Abelian Meissner effect.

At this point one may ask how realistic is the above theory of non-Abelian superconductivity. In particular, one may ask why the oppositely charged condensates should form a doublet. To answer this question we notice that  $\phi_+$  and  $\phi_-$  can always be put into a doublet (at least formally). So the real question is whether the interaction potential can be made  $SU(2)$  symmetric or not. The answer, of course, depends on materials. In fact we expect that the symmetry will be broken in real materials. Nevertheless one may still treat the  $SU(2)$  symmetry as an approximate symmetry. In this case one can adopt the above Landau-Ginzburg potential as a simplest potential in the first approximation, and treat the symmetry breaking interaction perturbatively in two-gap superconductor. In this sense we believe that the above theory of non-Abelian superconductivity could be able to describe the qualitative, if not quantitative, features of two-gap superconductor made of oppositely charged condensates.

This implies two things. An immediate implication is the existence of the non-Abelian magnetic vortex which has non-Abelian flux quantization in MgB<sub>2</sub>. The above discussion tells that a two-gap superconductor, regardless of the charge content of the doublet condensate, should allow the non-Abelian magnetic vortex. The other implication is that the non-Abelian dynamics could play a crucial role in condensed matter physics, in particular in multi-component condensed matter. Our result makes it clear that, implicitly or explicitly, the underlying dynamics of multi-component condensed matters can ultimately be related to a non-Abelian dynamics.

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